Polynomial Methods

Quantitative bounds for the Kakeya problem.
Joints Theorem for Tubes using Polynomials.

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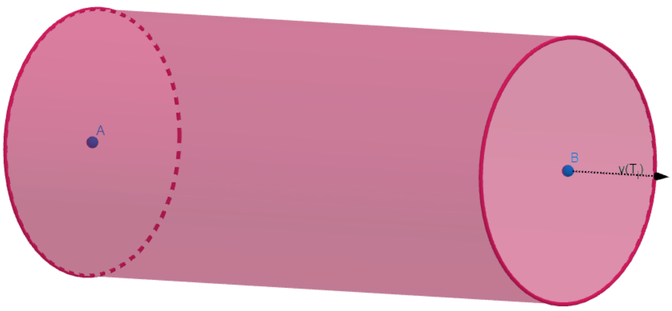
1 Besicovitch construction

Definition 1. *Kakeya set of Tubes*
Given a set $\{T_i\}_{i\in\mathbb{N}_{\leq M}} \subset \mathbb{R}^n$ of tubes of radius 1 and length N.
Supposed $\{\nu(T_i)\}_{i\in\mathbb{N}_{\leq M}}$ is $\frac{1}{N}$ separated and $\frac{2}{N}$ dense in $\mathcal{S}^{n-1}$.
Where $\mathcal{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and $\nu(T_i)$ is the unit vector parallel to the axis of symmetry of T.



Lemma 2. *Besicovitch construction*
Given a Kakeya set of tubes $\{T_i\}_{i\in\mathbb{N}_{\leq \bar{M}}}$ (with width 1 and length N) in $\mathbb{R}^2$ then $|\bigcup_{i=1}^{\bar{M}} T_i| > c\frac{N^2}{\log(N)}$ for some fixed $c > 0$.

Proof. Since $\{\nu(T_i)\}_{i\in\mathbb{N}_{\leq \bar{M}}}$ is $\frac{1}{N}$ separated and $\frac{2}{N}$ dense we can take all $\nu(T_i)$ with positive x and y coordinates and number the tubes s.t. $\frac{5|i-j|}{6N} \leq |\nu(T_i) - \nu(T_j)| \leq 2\frac{|i-j|}{N}$ (left inequality with 5).
By the computation from below we get for $\nu(T_i), \nu(T_j)$ with angle $\leq \pi/2$ that $|T_i \cap T_j| \leq 2\frac{N}{|i-j|}$.

Now we compute

$$\int_{\mathbb{R}^2} (\sum_{i=1}^{M} \mathbb{I}_{T_i})^2 = \int_{\mathbb{R}^2} \sum_{i,j=1}^{M} \mathbb{I}_{T_i}\mathbb{I}_{T_j} = \int_{\mathbb{R}^2} \sum_{i,j=1}^{M} \mathbb{I}_{T_i\cap T_j} \leq \int_{\mathbb{R}^2} \sum_{i=1}^{M} \mathbb{I}_{T_i} + \sum_{i,j=1,i\neq j}^{M} 2\frac{N}{|i-j|}$$

Since a Kakeya set of tubes is $1/N$ separated and $2/N$ dense, it is $\frac{5}{6}N\pi \leq \bar{M} \leq 2N\pi$ (by 4, 5) and therefore $\frac{5}{24}N\pi \leq M \leq \frac{1}{2}N\pi$. Thus

$$\int_{\mathbb{R}^2} \sum_{i=1}^{M} \mathbb{I}_{T_i} = M|T_1| \leq N^2\frac{\pi}{2} \text{ and}$$

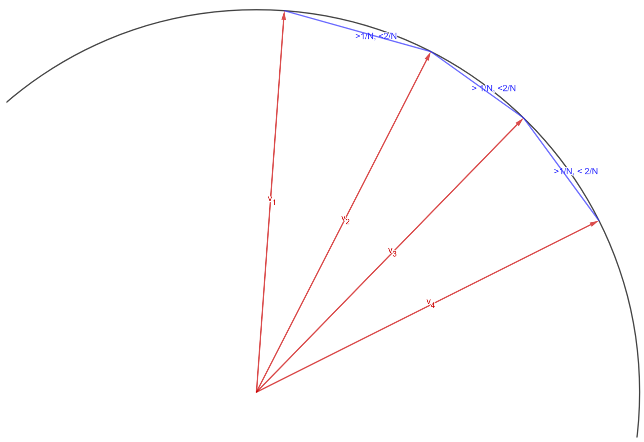


Figure 1: ordering of the unit vectors $\nu(T_i)$ belonging to the tubes

$$\sum_{i,j=1,i\neq j}^{M} 2\frac{N}{|i-j|} \leq 2\pi N \sum_{i=1}^{M} \frac{N}{i} \leq 2\pi N^2(\log(M)+1) \leq 3\pi N^2 \log(N)$$

$$\Rightarrow \int_{\mathbb{R}^2} \Big(\sum_{i=1}^{M} \mathbb{I}_{T_i}\Big)^2 \leq 10N^2 \log(N)$$

Let $K = \bigcup_{i=1}^{M} T_i$

It is $|T_i| = N \times 1$, thus since $\frac{5}{24}N\pi \leq M$: $\int_{\mathbb{R}^2} \sum_{i=1}^{M} \mathbb{I}_{T_i} \geq \frac{1}{2}N^2$

Now by Cauchy-Schwarz and the above inequality:

$$\frac{1}{2}N^2 \leq \int_{\mathbb{R}^2} \sum_{i=1}^{M} \mathbb{I}_{T_i} = \int_{\mathbb{R}^2} \sum_{i=1}^{M} \mathbb{I}_{T_i} \cdot \mathbb{I}_K \leq \Big(\int_{\mathbb{R}^2} \Big(\sum_{i=1}^{M} \mathbb{I}_{T_i}\Big)^2\Big)^{\frac{1}{2}} \cdot |K|^{\frac{1}{2}}$$

$$\leq \left(10N^2 \log(N)\right)^{\frac{1}{2}} \cdot |K|^{\frac{1}{2}} \Leftrightarrow |K| \geq \frac{1}{20^2}\frac{N^2}{\log(N)}$$

□

Lemma 3. *Computation for* $|T_i \cap T_j| \leq 2\frac{N}{|i-j|}$.

Proof. As one can see in the picture for two tubes T_i, T_j with infinite length which are not identical the volume of $T_i \cap T_j$ can be computed. This computation works as an upperbound for the case where the tubes have finite length.
The volume is given by the rhomboid with height 1 and one side u.
The length of u can be computed as $\frac{1}{\cos(\frac{\pi}{2}-\theta))}$ where $\theta < \pi/2$ is the angle between ν_i and ν_j $(\nu_i = \nu(T_i), \nu_j = \nu(T_j))$.
Further it is:

$$|\nu_i - \nu_j| = \sqrt{(1-\cos(\theta))^2 + \sin(\theta)^2} = \sqrt{2(1-\cos(\theta))}$$

For $\theta \leq \pi/2$, $\frac{\cos(\frac{\pi}{2}-\theta))}{\sqrt{2(1-\cos(\theta))}}$ is minimal for $\theta = \pi/2$ with it being $\frac{1}{\sqrt{2}}$.

It follows

$$|T_i \cap T_j| \leq \sqrt{2}\frac{1}{|\nu_i - \nu_j|} \leq 2\frac{N}{|i-j|}$$

□

Lemma 4. *Computation that for a Kakeya set of tubes the angle between each two tubes is* $\geq \frac{1}{N}$. *Especially if for two unit vectors* ν_i, ν_j, $K > 1$: $|\nu_i - \nu_j| > \frac{1}{K}$, *then the angle between these two unit vectors is greater than* $\frac{1}{K}$.

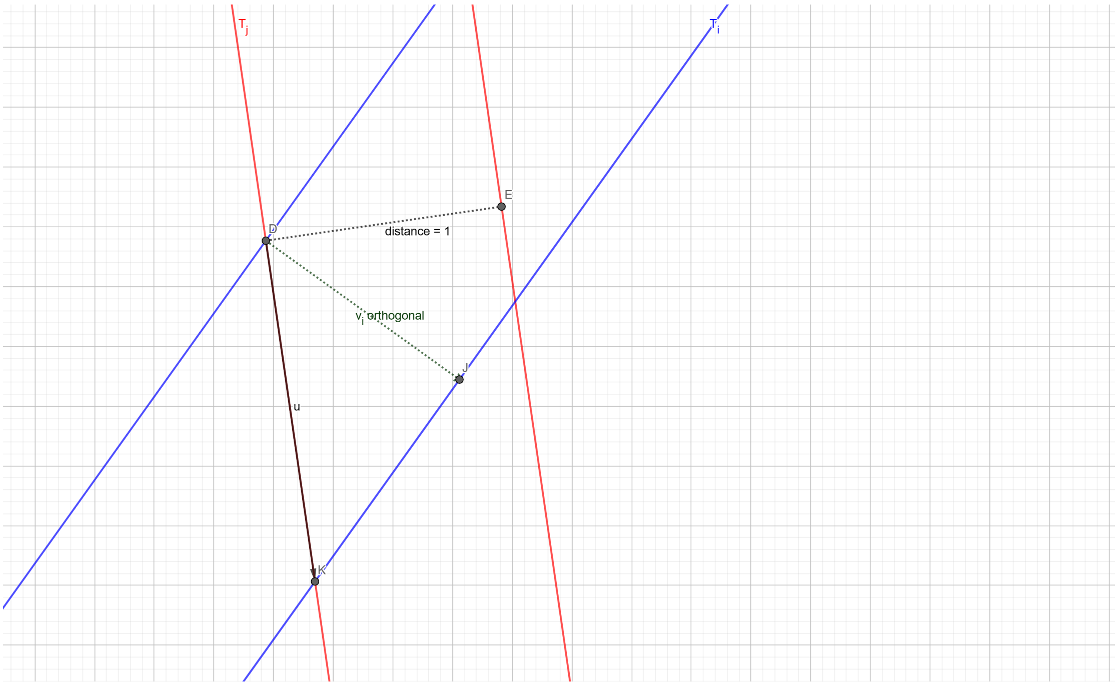


Figure 2: intersection $T_i \cap T_j$ for tubes T_i, T_j

Proof. Given two unit vector ν_i, ν_j with $|\nu_i - \nu_j| \geq \frac{1}{K}$.
Let θ the angle between ν_i and ν_j.

Then it is

$$\sqrt{(1-\cos(\theta))^2 + \sin(\theta)^2} \geq \frac{1}{K} \Leftrightarrow (1-\cos(\theta))^2 + \sin(\theta)^2 \geq \frac{1}{K^2}$$

$$\Leftrightarrow 2(1-\cos(\theta)) \geq \frac{1}{K^2} \Leftrightarrow \cos(\theta) \leq 1 - \frac{1}{2K^2}$$

since $\arccos : [-1,1] \to \mathbb{R}$ is monoton decreasing: $\Leftrightarrow \theta \geq \arccos(1 - \frac{1}{2K^2})$

It is $\frac{d}{dx} \arccos(1-x) = \frac{1}{\sqrt{2x-x^2}}$ and therefore

$$\arccos(1 - \frac{1}{2K^2}) = \arccos(1 - \frac{1}{2K^2}) - \arccos(1-0)$$

$$= \int_0^{\frac{1}{2K^2}} \frac{1}{\sqrt{2x-x^2}} dx \geq \int_0^{\frac{1}{2K^2}} \frac{1}{\sqrt{2x}} dx = \frac{1}{2K} \int_0^1 \frac{1}{\sqrt{x}} dx = \frac{1}{K}$$

□

Lemma 5. *Computation that if for two unit vectors ν_i, ν_j the angle θ between ν_i, ν_j is greater than $1/K$ then $|\nu_i - \nu_j| > \frac{5}{6K}$ for any $K > 1$.*

Proof. Given two unit vector ν_i, ν_j with $\theta \geq \frac{1}{K}$.
It is

$$\sin(\theta) \leq |\nu_i - \nu_j|, \sin(1/K) = \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{K})^{2k+1}}{(2k+1)!} \geq \frac{1}{K} - \frac{1}{6K^3} \geq \frac{5}{6K}$$

□

2 L^p-Kakeya conjecture

Let $\{T_i\}_{i \in \mathbb{N}_{\leq M}} \subset \mathbb{R}^n$ a Kakeya set of tubes and $\{T_i^0\}_{i \in \mathbb{N}_{\leq \bar{M}}} \subset \mathbb{R}^n$ a Kakeya set of tubes centered in the origin, it holds $\forall p > \frac{n}{n-1}$: $\exists C > 0$

$$\| \sum_{i=1}^{M} \mathbb{I}_{T_i} \|_{L^p} \leq C \| \sum_{i=1}^{\bar{M}} \mathbb{I}_{T_i^0} \|_{L^p}$$

3 Bush argument

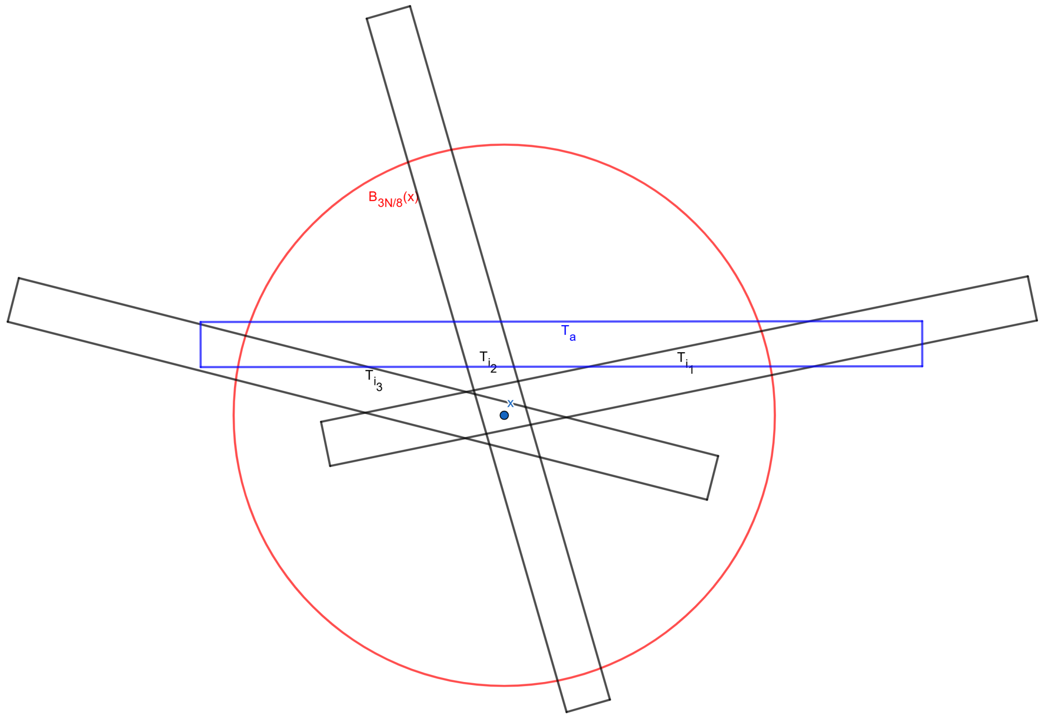


Figure 3: bush through x, red: circle of radius $3N/8$ around x, black: tubes which are in the bush through x, blue: tube which is not in the bush through x

Lemma 6. *(Bush argument) Given a Kakeya set of tubes* $\{T_i\}_{i\in\mathbb{N}_{\leq M}} \subset \mathbb{R}^n$*, then it is* $|\bigcup_{i=1}^{M} T_i| \geq c(n)N^{\frac{n+1}{2}}$ *for some* $c(n) > 0$ *which is fixed for given* n*.*

Proof. The volume of an n-dimensional tube is the volume of an (n-1)dimensional ball of radius 1 times N which is $\frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} \times N$

For a Kakeya set of tubes, we have a set of size at least $(\frac{2\pi N}{25})^{n-1}$ many tubes s.t. for each pair of tubes T_i, T_j: $|\nu(T_i) - \nu(T_j)| \geq \frac{8}{N}$ (see 14).
From now on we consider only this new set of tubes.

Let $K = \bigcup_{i=1}^{M} T_i$ then it exists $x \in K$ with x lying in at least $\sum_{i=1}^{M} \frac{|T_i|}{|K|} \geq b\frac{N^n}{|K|}$ $(b = (\frac{2\pi}{25})^{n-1}\frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)})$different tubes by the pigeon hole principle.

We call the set of tubes containing x the bush through x.
The bush through x is outside of $B_{3N/8}(x)$ partially disjoint since the distances between the $\nu(T_i), \nu(T_j)$ are greater than $8/N$ (see 15).

$|T_i \setminus B_{3N/8}(x)| \geq 1/4 \cdot |T_i| = 1/4\frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} \times N$

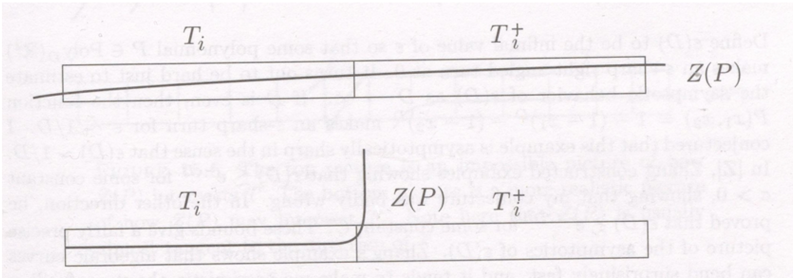
Thus the union of the tubes in the bush has volume $|K| \geq 1/4 \cdot |T_i| \cdot b\frac{N^n}{|K|} = 1/4\frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} \times N \times b\frac{N^n}{|K|} = c(n) \cdot \frac{N^{n+1}}{|K|}$ and since the bush of tubes through x is part of K: $\Rightarrow |K| \geq c(n) \cdot \frac{N^{n+1}}{|K|} \Leftrightarrow |K| \geq \sqrt{c(n)}N^{\frac{n+1}{2}}$ □

4 Difficulties of the Polynomial Method for the Kakeya Problem

Let $\{T_i\}_{i\in\mathbb{N}_{\leq M}}$ a Kakeya set of tubes in $\mathbb{R}^n$ and $K = \bigcup_{i=1}^{M} T_i$ with $C(n)N^{n-\gamma} \geq |K| \geq c(n)N^{n-\gamma}$, $C(n), c(n) > 0$
How can one imitate the proof from the finite field Kakeya problem?
1. Observation: There is no non zero polynomial P which is zero on K, since K contains an open ball in $\mathbb{R}^n$.

Idea take a unit lattice and let P vanish on every center of a unit cube intersecting a tube.
By parameter counting we could find P with degree $\leq k(n)N^{1-\frac{\gamma}{n}}$ for some $k(n) > 0$.
We would have $\sim N$ points in each tube where P vanishes, but since they do not have to lie on one line it is not clear how to deal with them.
$\Rightarrow$ Vanishing lemma is the key difficulty

Let T_i^+ be the translation of T_i by $N\nu(T_i)$.



Does P vanish at many evenly distributed points along T_i^+?
Unfortunately not necessary.

Let $D \in \mathbb{N}$ big, then the curve $y = x^D$ is close to the x-axis for $0 \leq x \leq 1 - D^{-1/2}$ but around $x = 1$ the curve turns sharply and is close to the $x = 1$ line.
It might be possible to show that an algebraic surface cannot bend too sharp in many points. But this question is very hard too solve.

4.1 Important Observations

Theorem 7. *Polynomial Ham Sandwich theorem*
Let $D \geq 1$, $m = \binom{n+D}{n} - 1$, $F_1, ...F_m$ m bounded sets in $\mathbb{R}^n$, then it exists a polynomial P of degree at most D which bisects $F_i \, \forall 1 \leq i \leq m$. https://n.ethz.ch/~ywigderson/math/teaching/HamSandwichNotes.pdf

Lemma 8. *Let $T_i \in \mathbb{R}^n$ a tube, D_i a cross section of T_i and for $x \in D_i$ ℓ_x the line through x orthogonal to D_i, $P \neq 0$ a polynomial in $\mathbb{R}^n$.*
Then $|Z(P) \cap \ell_x| \leq degP$ for almost all $x \in D_i$.

Consider again the unit lattice and denote with $\mathcal{Q}(K)$ all cubes intersecting K and with $\mathcal{Q}(T_i) \subset \mathcal{Q}(K)$ all cubes intersecting the tube T_i.

Let P a minimal polynomial bisecting each cube in $\mathcal{Q}(K)$.
As we have seen $|\ell_x \cap Z(P)| \leq degP \leq k(n)N^{1-\frac{\gamma}{n}}$ (second inequality: Polynomial Ham Sandwich theorem) for almost all $x \in D_i$.
Therefore for the average over all cubes intersecting T_i and almost all $x \in D_i$
$AVG_{Q\in\mathcal{Q}(T_i)}|\ell_x \cap Z(P) \cap Q| \leq K(n)N^{-\frac{\gamma}{n}}$ for some $K(n) > 0$,
also $AVG_{Q\in\mathcal{Q}(T_i), x\in D_i}|(\ell_x \cap Z(P) \cap Q| \leq K(n)N^{-\frac{\gamma}{n}} \ll 1$ which is only possible if $Z(P) \cap Q$ is approximately parallel to T_i for most cubes .

$\Rightarrow$ structure called planiness i.e. for each cube there is a hyperplane $\pi(Q)$ s.t. all tubes intersecting Q are almost tangent to $\pi(Q)$.

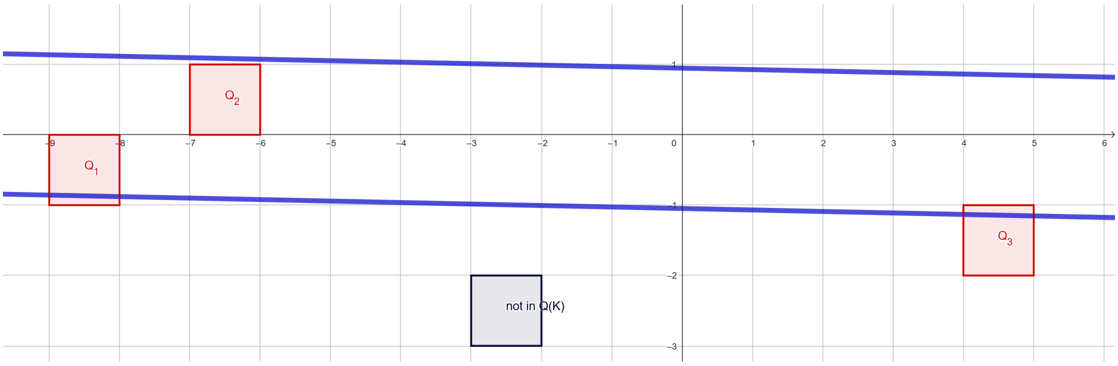


Figure 4: Examples of cubes intersecting a tube (blue lines) and therefore are contained in Q(K) and which are not

5 Joints theorem for tubes

Theorem 9. *Joints theorem for Tubes*
Given a set of lines $\{\ell_{j,a}\} \subset \mathbb{R}^n$, $1 \leq j \leq n$, $1 \leq a \leq A$ *s.t. each line* $\ell_{j,a}$ *has an angle of at most* $\frac{1}{100n}$ *with the* x_j *axis, with* $T_{j,a}$ *the infinite cylinder of radius 1 centered around* $\ell_{j,a}$.
Then the volume of $I = \bigcap\limits_{j=1}^{n} \bigcup\limits_{a=1}^{A} T_{j,a}$ *is smaller than* $c(n)A^{\frac{n}{n-1}}$ *for some* $c(n) > 0$.

Definition 10. *Directed volume of a surface*
Let $\mathcal{M}$ *a smooth hypersurface in* $\mathbb{R}^n$, *let for each* $x \in \mathcal{M}$ $N(x)$ *a unit normal vector to* $\mathcal{M}$ *at the point* x.
For $\nu \in \mathcal{S}^{n-1}$ *the directed volume of* $\mathcal{M}$ *perpendicular to* ν *is defined as:*
$V_{\mathcal{M}}(\nu) = \int\limits_{\mathcal{M}} |N(x) \cdot \nu| d\mathcal{H}^{n-1}(x)$

Lemma 11. *For* $\mathcal{M}$ *a smooth hypersurface,* $\nu \in \mathcal{S}^{n-1}$ *it is* $V_{\mathcal{M}}(\nu) = \int\limits_{\nu^{\perp}} |\mathcal{M} \cap \pi^{-1}(y)| d\mathcal{H}^{n-1}(y)$ *where* π *is the orthogonal projection from* $\mathbb{R}^n$ *to* $\nu^{\perp}$.

Lemma 12. *Given a smooth hypersurface* $\mathcal{M} \subset \mathbb{R}^n$, *unit vectors* $\nu_1, ...\nu_n$ *with* ν_i *having an angle of at most* $\frac{1}{100n}$ *with the* x_i *axis then* $\mathcal{H}^{n-1}(\mathcal{M}) \leq 2 \sum\limits_{i=1}^{n} V_{\mathcal{M}}(\nu_i)$.

Proof. We show that for any given point $x \in \mathcal{M}$ it is $\sum\limits_{i=1}^{n} |N(x) \cdot \nu_i| \geq \frac{1}{2}$.
Let $e_1, ...e_n$ the coordinate vectors, it is $\sum\limits_{i=1}^{n} |N(x)e_i| \geq \sum\limits_{i=1}^{n} |N(x)e_i|^2$ (scalar product of two unit vector is ≤ 1)
And $\sum\limits_{i=1}^{n} |N(x)e_i|^2 = 1$ by Parseval's identity. Using 4 in the second inequality:

$$\sum_{i=1}^{n} |N(x)\nu_i| \geq \sum_{i=1}^{n} |N(x)e_i| - \sum_{i=1}^{n} |N(x)||(e_i - \nu_i)| \geq 1 - \sum_{i=1}^{n} \frac{1}{100n} = 1 - \frac{1}{100} \geq \frac{1}{2}$$

$$\Rightarrow 2\sum_{i=1}^{n} V_{\mathcal{M}}(\nu_i) = 2\int\limits_{\mathcal{M}} \sum_{i=1}^{n} |N(x) \cdot \nu_i| d\mathcal{H}^{n-1}(x) \geq 2\int\limits_{\mathcal{M}} \frac{1}{2} d\mathcal{H}^{n-1}(x) = \mathcal{H}^{n-1}(\mathcal{M})$$

□

Lemma 13. *Cylinder estimate*
Let T *be an infinite cylinder in* $\mathbb{R}^n$ *of radius* r, ν *a unit vector parallel to the axis of symmetry of* T, $Z(P) = \{x \in \mathbb{R}^n \,|\, P(x) = 0\}$ *i.e. the vanishing set of* P.
Then $V_{Z(P) \cap T}(\nu) \leq c(n)r^{n-1}deg(P)$, $c(n) > 0$

Proof. Let π the projection from T to $\nu^{\perp} \cap T$ which is a $n-1$ dimensional ball.
For almost all $y \in \nu^{\perp} \cap T$: $|\pi^{-1}(y) \cap Z(P)| \leq deg(P)$ (lemma 8).

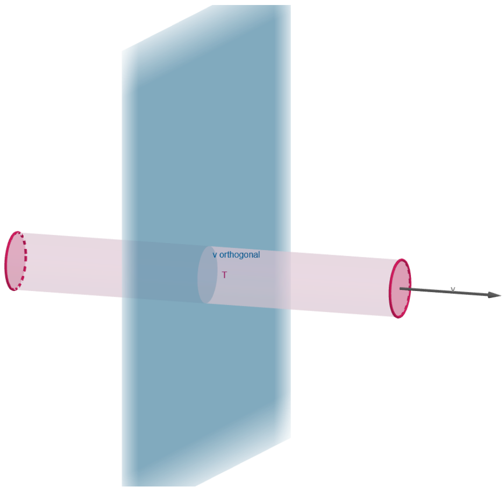


Figure 5: Cylinder estimate: tube T, orthogonal complement of ν (blue plane)

Then by lemma 11 for the first equation:

$$V_{Z(P)\cap T}(\nu) = \int_{\nu^\perp} |Z(P) \cap T \cap \pi^{-1}(y)| d\mathcal{H}^{n-1}(y) \leq \int_{\nu^\perp \cap T} deg(P) d\mathcal{H}^{n-1}(y) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} r^{n-1} deg(P)$$

□

Proof. Joints theorem for tubes
Consider the unit cubical lattice, let $\{Q_1, ... Q_V\} = \mathcal{Q}(K)$ all unit cubes which intersect I.
To show: $V \leq c(n) A^{\frac{n}{n-1}}$ and since $|Q_i \cap I| \leq 1$ ($|Q_i| = 1$), $I \subset \bigcup_{i=1}^{V} Q_i$ this is enough to show.

Let P a non-zero polynomial s.t. $Z(P) = \{x \in \mathbb{R}^n \,|\, P(x) = 0\}$ bisects each cube $Q_1, ... Q_V$ (which exists by the Polynomial Ham Sandwich Thereom) and s.t. $deg(P) \leq k(n) V^{\frac{1}{n}}$ for some $k(n) > 0$

Each cube $Q_1, ... Q_V$ intersects I and therefore there is a tube $T_{j,a}$ for each $1 \leq j \leq n$ s.t. Q_i intersects $T_{j,a}$.
Let for each j $T_j(Q_i)$ one of the tubes $T_{j,a}$ $1 \leq a \leq A$(there is at least one) which intersects Q_i with direction $\nu_{j,i}$.

By lemma 12 we get

$$\sum_{j=1}^{n} V_{Z(P)\cap Q_i}(\nu_{j,i}) \geq \frac{1}{2} \mathcal{H}^{n-1}(Z(P) \cap Q_i) \geq \frac{1}{2}$$

Therefore $\exists \nu_{j,i}$ with $V_{Z(P)\cap Q_i}(\nu_{j,i}) \geq \frac{1}{2n}$.
We assign the cube Q_i to the tube belonging to this vector $T_j(Q_i)$.
Since we assigned V cubes to nA tubes we get one tube with $\frac{V}{nA}$ cubes assigned to it, we denote it by T and it's direction by ν.

Now let $\bar{T}$ the cylinder with radius 2n and central axis ν, each of the $\frac{V}{nA}$ cubes Q_i lies in $\bar{T}$, since $\max_{x,y \in Q_i} |x - y| \leq \sqrt{n}$ and Q_i intersects T.

We get $\frac{V}{2n^2 A} \leq V_{Z(P)\cap \bar{T}}(\nu)$ (since$V_{Z(P)\cap Q_i}(\nu_{j,i}) \geq \frac{1}{2n}$) , by the cylinder estimate $V_{Z(P)\cap \bar{T}}(\nu) \leq C(n) V^{\frac{1}{n}}$ and therefore $V \leq (2n^2 C(n))^{\frac{n}{n-1}} A^{\frac{n}{n-1}}$ □

6 Additional Computations

Lemma 14. *Number of tubes in a Kakeya set*
Given a Kakeya set of tubes, we have at least $(\frac{5N\pi}{6})^{n-1}$ *many tubes in this set.*

Further we have a subset M of at least $(\frac{3N\pi}{25})^{n-1}$ many tubes s.t. for each T_i, $T_j \in M$: $|\nu(T_i) - \nu(T_j)| > \frac{8}{N}$.

Proof. It needs to hold that the set is $2/N$ dense in $\mathcal{S}^{n-1}$. Consider the polar coordinate representation of the vectors in $\mathcal{S}^{n-1}$ (i.e. $\nu_i = (\theta_1^i, ...\theta_{n-1}^i)$).
By 5 we know that if the angle between $\nu(T_i)$ and $\nu(T_j)$ is greater than $\frac{12}{5N}$ then $|\nu(T_i) - \nu(T_j)| \geq \frac{2}{N}$.
Therefore if we can find for some $\nu(T_i)$ one angle θ_l^i of $\nu(T_i)$ s.t. $\forall j \ |\theta_l^i - \theta_l^j| > \frac{12}{5N}$ where θ_l^j is the l.th angle of $\nu(T_j)$ then the set cannot be a Kakeya set of tubes.
We have n-1 angles, thus we get at least $(\frac{2\pi}{\frac{12}{5N}})^{n-1} = (\frac{5N\pi}{6})^{n-1}$ tubes.

Consider again the coordinate representation of the unit vectors

If we have some $\nu(T_i), \nu(T_j)$ s.t. for some angle $|\theta_l^i - \theta_l^j| > 8\frac{6}{5N}$ then $|\nu(T_i) - \nu(T_j)| > \frac{8}{N}$.
We now divide each polar coordinate into $\frac{2N\pi}{25}$ intervals, in each of these intervals $[\frac{25i}{N}, \frac{25(i+1)}{N}]$ there has to be some vector $\nu(T_i)$ in the middle interval $[\frac{25i+10}{N}, \frac{25i+15}{N}]$ because the Kakeya set is $2/N$ separated and an angle distance of $> \frac{12}{5N}$ correlates with a distance of $> 2/N$.

For all tubes $\nu(T_j)$ which are not in the interval $[\frac{25i}{N}, \frac{25(i+1)}{N}]$
$|\theta_l^j - \theta_i^l| \geq \frac{10}{N} > \frac{48}{5N}$ and therefore $|\nu(T_j) - \nu(T_i)| \geq 8/N$.

We get a subset M of at least $(\frac{2\pi}{\frac{25}{N}})^{n-1} = (\frac{2N\pi}{25})^{n-1}$ tubes with the condition that $\forall T_i, T_j \in M$: $|\nu(T_i) - \nu(T_j)| > \frac{8}{N}$. □

Lemma 15. *Given $x \in \mathbb{R}^n$ and a set M of tubes s.t. for each $T_i \in M$: $x \in T_i$ and s.t. $\forall T_i, T_j \in M$: $|\nu(T_i) - \nu(T_j)| > \frac{8}{N}$, then the tubes in M are disjoint outside of $B_{3N/8}(x)$.*

Proof. Let $T_i \in M$, $y \in T_i$, $y \notin B_{3N/8}(x) \Rightarrow \exists L > 3N/8 \ y \in B_1(x + L\nu(T_i))$

Supposed $y \in T_j$ for some $T_j \in M$, then we need to have $\tilde{L}$ s.t. $y \in B_1(x + \tilde{L}\nu(T_j))$ and thus $|L\nu(T_i) - \tilde{L}\nu(T_j)| \leq 2$.

Let θ the angle between $\nu(T_i)$ and $\nu(T_j)$.

$$|L\nu(T_i) - \tilde{L}\nu(T_j)| = \sqrt{(L - \tilde{L}\cos(\theta))^2 + \tilde{L}^2 sin(\theta)^2} = \sqrt{L^2 + \tilde{L}^2 - 2\cos(\theta)L\tilde{L}}$$

$$\geq \sqrt{L^2 + \tilde{L}^2 - 2\cos\Big(\frac{8}{N}\Big)L\tilde{L}} \geq \sqrt{\Big(1 - \frac{32}{N^2}\Big)(L - \tilde{L})^2 + \frac{32}{N^2}(L^2 + \tilde{L}^2)}$$

which is:

$$\geq \sqrt{4\Big(1 - \frac{1}{2}\Big) + \frac{32}{N^2}\Big(\frac{3N}{8}\Big)^2} = \sqrt{2 + \frac{9}{2}} > 2$$

for $|L - \tilde{L}| \geq 2$ and $N > 8$

and for $|L - \tilde{L}| < 2$ and $N > 16$:

$$\geq \sqrt{\frac{32}{N^2}\Big(\Big(\frac{3N}{8}\Big)^2 + \Big(\frac{2N}{8}\Big)^2\Big)} = \sqrt{\frac{9}{2} + 2} > 2$$

So for large N, $y \notin B_{3N/8}(x)$, $y \in T_i \Rightarrow y \notin T_j$ for $i \neq j$. □

References

[1] Larry Guth. *Polynomial Methods in Combinatorics*. 2016. ISBN: 978-1-4704-2890-7.